

SIMPLICITY OF ALGEBRAS ASSOCIATED TO ÉTALE GROUPOIDS

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ABSTRACT. We prove that the C^* -algebra of a second-countable, étale, amenable groupoid is simple if and only if the groupoid is topologically principal and minimal. We also show that if G has totally disconnected unit space, then the associated complex $*$ -algebra introduced by Steinberg is simple if and only if the interior of the isotropy subgroupoid of G is equal to the unit space and G is minimal.

1. INTRODUCTION

It has been known since Renault's seminal work in 1980 [27] that if a second-countable, étale, amenable groupoid is both minimal and topologically principal, then its C^* -algebra is simple. Whether the converse holds was an open question until now. In this paper we answer this question in the affirmative: the C^* -algebra of a second-countable, étale, amenable groupoid is simple if and only if the groupoid is both topologically principal and minimal. We also establish the corresponding theorem for the algebras $A(G)$ of [33]. There are many classes of C^* -algebras with étale groupoid models (see for example [8, 10, 12, 13, 15, 18, 25, 27, 30, 35]), so we expect that our results will find numerous applications.

Let G be a groupoid which is étale in the sense that $r, s : G \rightarrow G^{(0)}$ are local homeomorphisms. Arguments of [27] show that if G is *topologically principal*¹ in the sense that the units with trivial isotropy are dense in the unit space, and *minimal* in the sense that the unit space has no nontrivial open invariant subsets, then the reduced C^* -algebra $C_r^*(G)$ is simple. The converse of this theorem is false. It is true that if $C_r^*(G)$ is simple then G is minimal, but G need not be topologically principal (see Example 6.1).

If we re-interpret Renault's result as the assertion that for groupoids G whose full and reduced C^* -algebras coincide, if G is both topologically principal and minimal, then $C^*(G)$ is simple, the situation regarding the converse is less clear. Recent results about higher-rank graphs [20, 32] show that the converse *does* hold for some known classes of examples.

Complex algebras $A(G)$ associated to locally compact, Hausdorff, étale groupoids G with totally disconnected unit spaces were introduced recently by Steinberg in [33] and further examined in [7]. These algebras, which we call Steinberg algebras, include the complex Kumjian-Pask algebras of higher-rank graphs studied in [4], and hence the Leavitt path algebras of directed graphs studied in [1]. The criteria of [32] which characterise

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¹See Remark 2.3 regarding terminology.

simplicity of a k -graph C^* -algebra also characterise simplicity of the associated Kumjian-Pask algebra [4, Theorem 5.14]. Encouraged by this, we set out to investigate simplicity of $A(G)$.

We originally hoped to prove that G is topologically principal and minimal if and only if $A(G)$ is simple (amenability issues vanish in the algebraic setting). The literature suggested that it would be difficult to pinpoint a necessary and sufficient condition for simplicity of $C^*(G)$, but we hoped that many technical obstacles would vanish in the algebraic setting. Specifically, we expected that Renault's argument could be adapted to prove the "only if" implication; and we hoped that in the situation of algebras, where there are no continuity hypotheses to check when constructing representations, we could adapt the ideas of [32, Proposition 3.5] to prove the converse.

Our early attempts to prove the result failed, and eventually we came to the conclusion that the natural necessary condition was not that G is topologically principal, but that no open subset of $G \setminus G^{(0)}$ consists entirely of isotropy. For if this condition fails in an étale groupoid with totally disconnected unit space, then there exists a compact open set B consisting purely of isotropy on which the range and source maps are homeomorphisms. It follows that $1_{r(B)} - 1_B$ belongs to $A(G)$ and vanishes under a natural homomorphism from $A(G)$ to the algebra of endomorphisms of the free complex module $\mathbb{F}(G^{(0)})$ with basis $G^{(0)}$ (see Proposition 4.4). The condition that no open subset of $G \setminus G^{(0)}$ consists entirely of isotropy is equivalent to saying that the interior of the isotropy subgroupoid of G is $G^{(0)}$, and is formally weaker than G being topologically principal. What came as a bit of a surprise to us was that this weaker necessary condition, together with minimality, is also sufficient for simplicity of $A(G)$ (see Theorem 4.1): Renault's arguments run almost unchanged.

It came as a substantially greater surprise when we discovered that the arguments we had developed for $A(G)$ could be adapted to the C^* -algebraic setting provided that G is second-countable, and that we could then drop the requirement that $G^{(0)}$ is totally disconnected. That is, we had discovered the aforementioned necessary and sufficient condition for simplicity of $C^*(G)$: if G is second-countable, étale, and amenable, then $C^*(G)$ is simple if and only if the interior of the isotropy subgroupoid of G is $G^{(0)}$ and G is minimal.

This led us back to our original question, rephrased as follows: is an étale groupoid G topologically principal if and only if the interior of the isotropy is equal to $G^{(0)}$? The answer is "no" in general (see Example 6.3), but if G is second-countable and étale, then the answer is "yes" (see Lemma 3.3). So, a little ironically, the result we had hoped to prove about $A(G)$ is not true in general, but the corresponding result for $C^*(G)$, which we originally had no notion of proving, is (see Theorem 5.1).

After a short preliminaries section, we describe in Section 3 a number of equivalent conditions on a locally compact, Hausdorff, étale groupoid G , one of which is that the interior of the isotropy subgroupoid of G is $G^{(0)}$. Next, we show that these equivalent conditions are formally weaker than G being topologically principal, but are equivalent to G being topologically principal if G is second countable. We present our structure theorems for the Steinberg algebra $A(G)$ in Section 4. In Section 5 we prove C^* -algebraic versions of these results. We choose to pay the price of more-technical statements in order to describe how our techniques apply to non-amenable groupoids. In a short examples section we indicate why our techniques cannot be adapted to characterise simplicity of

the reduced C^* -algebra of an étale groupoid and why our results do not extend readily to twisted groupoid C^* -algebras. We also provide an example of a non-étale groupoid in which every unit has infinite isotropy but no open set consists entirely of isotropy. Moreover, by changing the topology, we construct an étale groupoid with totally disconnected unit space (which is not second countable) with the same property. We finish by discussing how our results relate to those of Exel-Vershik [11] and of Exel-Renault [10].

In the late stages of the preparation of this article we learned that Renault has proved a version of Lemma 3.3 in [30, Proposition 3.6]. We thank Alex Kumjian for bringing this to our attention.

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Aidan thanks his Dad for making him learn his Baire category theory. Perhaps you were right about eating vegetables too...

2. PRELIMINARIES

If X is a topological space and $D \subseteq X$, then we shall write D° for the interior $\bigcup\{U \subseteq D : U \text{ is open in } X\}$ of D .

A groupoid G is a small category in which every morphism has an inverse. When G is endowed with a topology under which the range, source, and composition maps are continuous, G is called a topological groupoid. We say G is *étale* if r and s are local homeomorphisms. It then follows that $G^{(0)} := \{\gamma\gamma^{-1} : \gamma \in G\}$ is both open and closed in G . For a more detailed description of étale groupoids, see [23].

A subset B of G such that r and s both restrict to homeomorphisms of B is called a *bisection* of G . If G is a locally compact, Hausdorff, étale groupoid, then there is a base for the topology on G consisting of precompact open bisections. As demonstrated in [7, 33], if $G^{(0)}$ is totally disconnected and G is locally compact, Hausdorff, and étale, then there is base for the topology on G consisting of compact open bisections.

For subsets D, E of $G^{(0)}$, define

$$G_D := \{\gamma \in G : s(\gamma) \in D\}, \quad G^E := \{\gamma \in G : r(\gamma) \in E\} \quad \text{and} \quad G_D^E := G^E \cap G_D.$$

In a slight abuse of notation, for $u, v \in G^{(0)}$ we denote $G_u := G_{\{u\}}$, $G^v := G^{\{v\}}$ and $G_u^v := G^v \cap G_u$. The *isotropy group* at a unit u of G is the group $G_u^u = \{\gamma \in G : r(\gamma) = s(\gamma) = u\}$. We say u has trivial isotropy if $G_u^u = \{u\}$. The *isotropy subgroupoid* of a groupoid G is $\text{Iso}(G) := \bigcup_{u \in G^{(0)}} G_u^u$. Since r and s are continuous, the isotropy subgroupoid of G is a closed subset of G .

A subset D of $G^{(0)}$ is called *invariant* if $s(\gamma) \in D \implies r(\gamma) \in D$ for all $\gamma \in G$. Since G contains inverses, this is equivalent to saying that $D = \{r(\gamma) : s(\gamma) \in D\} = \{s(\gamma) : r(\gamma) \in D\}$; hence $G_D = G^D$, and G_D is a groupoid with unit space D . Also, D is invariant if and only if its complement is invariant.

For subsets S and T of G , define $ST = \{\gamma\alpha : \gamma \in S, \alpha \in T, \text{ and } s(\gamma) = r(\alpha)\}$. We write Su for $S\{u\}$.

Definition 2.1. Let G be a locally compact, Hausdorff groupoid. We say that G is *topologically principal* if $\{u \in G^{(0)} : G_u^u = \{u\}\}$ is dense in $G^{(0)}$. We say that G is *minimal* if $G^{(0)}$ has no nontrivial open invariant subsets.

Remark 2.2. To relate our later results to those of Thomsen [34], we observe that an étale Hausdorff groupoid G is topologically principal if and only if each open invariant subset of $G^{(0)}$ contains a point with trivial isotropy. To see this, note that the “only if” implication is trivial. So suppose that every open invariant set contains a point with trivial isotropy, and fix an open subset U of $G^{(0)}$. Then $r(G_U)$ is an open invariant set, so contains a point u with trivial isotropy. Fix $\gamma \in G_U$ with $r(\gamma) = u$. Since $G_{s(\gamma)}^{s(\gamma)} = \gamma^{-1}G_u^u\gamma = \gamma^{-1}\{u\}\gamma = r(\gamma)$, we see that $s(\gamma)$ has trivial isotropy. That is, the set U contains a point with trivial isotropy. So G is topologically principal.

It follows immediately from this that if a minimal groupoid G has a unit with trivial isotropy then it is topologically principal.

Remark 2.3. In groupoid literature, the condition which we are calling *topologically principal* has gone under this name and a number of others, including “essentially free,” “topologically free,” and “essentially principal.” We have chosen the one we believe to be least open to misinterpretation. The usage of the term “principal” for groupoids with everywhere-trivial isotropy is standard, so “topologically principal” is suggestive. We chose this term over the term “topologically free” because the latter has a different usage for transformation groups. The word “essentially” in the other two variants is more suggestive of a measure-theoretic rather than a topological condition. Our choice also seems to match what Renault himself has settled on [30, 31].

Similarly, our usage of the term *minimal* seems to be standard (see, for example, [27, Definition I.4.1]) but is by no means universal. In statements of theorems, we spell out what we mean by each of these conditions in an effort to avoid any misunderstandings.

3. TOPOLOGICALLY PRINCIPAL GROUPOIDS

In this section, we establish some structure results for étale groupoids. The following lemma establishes the equivalent conditions that we use in Theorem 4.1 to characterise simplicity of $A(G)$.

Lemma 3.1. *Let G be a locally compact, Hausdorff, étale groupoid. The following are equivalent:*

- (1) *The interior of the isotropy subgroupoid of G is $G^{(0)}$;*
- (2) *$\text{Iso}(G) \setminus G^{(0)}$ has empty interior;*
- (3) *for every nonempty open bisection $B \subseteq G \setminus G^{(0)}$, there exists $\gamma \in B$ such that $s(\gamma) \neq r(\gamma)$;*
- (4) *for every compact $K \subseteq G \setminus G^{(0)}$ and every open $U \subseteq G^{(0)}$, there exists an open subset $V \subseteq U$ such that $VKV = \emptyset$.*

Proof of Lemma 3.1. Since G is étale, $G^{(0)}$ is both open and closed in G . So the interior S° of any subset S of G is equal to the disjoint union $(S \cap G^{(0)})^\circ \cup (S \setminus G^{(0)})^\circ$. Thus (1) is equivalent to (2).

We have (2) \implies (3) because open bisections are in particular open sets. That G is étale also implies that the collection of all open bisections of G form a base for the topology on G . In particular, every open set contains an open bisection, giving (3) \implies (2).

To see (4) implies (3), we prove the contrapositive. Suppose that (3) does not hold, and fix an open bisection $B_0 \subseteq G \setminus G^{(0)}$ such that $r(\gamma) = s(\gamma)$ for all $\gamma \in B_0$. That is, $B_0 \subseteq \text{Iso}(G)$. By shrinking if necessary, we may assume that B_0 is precompact. Since G is locally compact and Hausdorff, G is regular. Thus, there is an open subset B of B_0

whose closure K is compact and contained in B_0 . Let $U = r(B)$, and fix a nonempty open subset V of U . Since $K \subseteq \text{Iso}(G)$, we have $VK = KV$, and in particular $VKV \neq \emptyset$. Hence (4) does not hold.

It remains to show that (3) implies (4). We begin with a claim.

Claim 3.2. Suppose that $B \subseteq G \setminus G^{(0)}$ is an open bisection and that $\gamma \in B \setminus \text{Iso}(G)$. Then there is an open set $V \subseteq r(B)$ such that $\gamma \in VB$ and $s(VB) \cap V = \emptyset$.

Proof of Claim 3.2. Since $r(\gamma) \neq s(\gamma)$ and G is Hausdorff, there exist open neighbourhoods W of $r(\gamma)$ and W' of $s(\gamma)$ such that $W \cap W' = \emptyset$. Let $V := W \cap r(BW')$. Notice that $r(\gamma) \in V$ so V is not empty. Then $\gamma \in VB$, and since B is a bisection, $s(VB) = s(WB \cap BW') \subseteq W'$ and hence is disjoint from $V \subseteq W$. \square Claim 3.2

Now suppose (3), and fix a compact $K \subseteq G \setminus G^{(0)}$ and an open $U \subseteq G^{(0)}$. We construct a nonempty open set $V \subseteq U$ such that $VKV = \emptyset$. If U is not a subset of $r(K)$, then $V = U \setminus r(K)$ will suffice, so suppose that $U \subseteq r(K)$. Because G is regular and $G^{(0)}$ is open, there is a base for the topology on $G \setminus G^{(0)}$ consisting of precompact open bisections whose closures are themselves contained in open bisections which do not intersect $G^{(0)}$. Since K is compact, we may cover K by a finite set \mathcal{B} of such precompact open bisections. For each $B \in \mathcal{B}$, fix an open bisection C_B such that $\overline{B} \subseteq C_B \subseteq G \setminus G^{(0)}$. For each $B \in \mathcal{B}$ the set UC_B is an open bisection which does not intersect $G^{(0)}$, and the $r(B)$ cover U so at least one UC_B is nonempty. So (3) implies that there exists $\gamma \in \bigcup_{B \in \mathcal{B}} UC_B \setminus \text{Iso}(G)$. Let $\mathcal{F} := \{B \in \mathcal{B} : \gamma \in UC_B\}$. For each $B \in \mathcal{F}$, Claim 3.2 yields an open set $V_B \subseteq r(UC_B)$ such that $s(V_B C_B) \cap V_B = \emptyset$. Let

$$V := U \cap \left(\bigcap \{V_B : B \in \mathcal{F}\} \right) \setminus \left(\bigcup \{\overline{r(B')} : B' \in \mathcal{B} \setminus \mathcal{F}\} \right).$$

Then V is open by definition, and nonempty because it contains $r(\gamma)$.

Fix $\alpha \in VK$; we must show $s(\alpha) \notin V$. Since $\alpha \in K$ and \mathcal{B} is a cover of K , we have $\alpha \in B$ for some $B \in \mathcal{B}$. Also, since $r(\alpha) \in V$, we have $B \in \mathcal{F}$. Hence

$$s(\alpha) \in s(VB) \subseteq s(V_B B),$$

and $s(VB) \cap V \subseteq s(V_B B) \cap V_B = \emptyset$. Therefore $s(\alpha) \notin V$. Thus (3) implies (4). \square

Lemma 3.3. *Let G be a locally compact, Hausdorff, étale groupoid. If G is topologically principal in the sense that the units with trivial isotropy are dense in $G^{(0)}$, then G satisfies the equivalent conditions described in Lemma 3.1. If G is second countable, then G is topologically principal if and only if it satisfies the equivalent conditions of Lemma 3.1.*

Proof. First suppose that G is topologically principal. Then [2, Lemma 2.3], implies that G satisfies Condition (4) of Lemma 3.1.

Now suppose that G is second-countable and satisfies Condition (3) of Lemma 3.1. Let U be the interior of the set of points in $G^{(0)}$ with nontrivial isotropy. We must show that U is empty. As in Remark 2.2, if $s(\gamma) \in U$, then $r(\gamma)$ has nontrivial isotropy. Hence $r(G_U)$ is open and consists of points with nontrivial isotropy, and so is contained in U ; since the reverse containment is trivial, it follows that U is an open invariant set. Thus $H := G_U$ is a second-countable, locally compact, Hausdorff, étale groupoid in which every unit has nontrivial isotropy, and $H^{(0)} = U$. We must show that $U = \emptyset$.

Recall that a subset S of a topological space X is *nowhere-dense* if the interior of the closure of S in X is empty. Fix an open bisection B in $H \setminus U$. Let $B_{\text{Iso}} := \{\gamma \in B :$

$r(\gamma) = s(\gamma)\}$. We claim that $r(B_{\text{Iso}})$ is nowhere-dense in U . To see this, suppose that $V \subseteq \overline{r(B_{\text{Iso}})}$ is open. Then VB is an open subset of B , and hence an open bisection. Fix $\gamma \in VB$. Then $r(\gamma) \in V$ can be written as $\lim_{n \rightarrow \infty} v_n$ for some sequence $v_n \in r(B_{\text{Iso}})$. For each n , let γ_n be the unique element of $v_n B$. Since $r|_B$ is a homeomorphism, we have $\gamma = \lim_{n \rightarrow \infty} \gamma_n$. Hence $s(\gamma) = \lim_{n \rightarrow \infty} s(\gamma_n) = \lim_{n \rightarrow \infty} r(\gamma_n) = r(\gamma)$. Now condition (3) of Lemma 3.1 implies that VB is empty, and hence V is empty.

Since H is second-countable and $H \setminus U$ is open, there is a countable collection \mathcal{B} of open bisections in H such that $H \setminus U = \bigcup \mathcal{B}$. Since $H_u^u \neq \{u\}$ for all $u \in U$, we have $\bigcup_{B \in \mathcal{B}} r(B_{\text{Iso}}) = U$. So U is a locally compact, Hausdorff space which can be expressed as a countable union of nowhere-dense sets. Hence the formulation of the Baire Category Theorem given in [16, Theorem 6.34] implies that U is empty. \square

Remark 3.4. The second statement of Lemma 3.3 need not hold if G is not second-countable (see Example 6.4) or if G is not étale (see Example 6.3).

4. SIMPLICITY OF STEINBERG ALGEBRAS

In this section, we restrict our attention to groupoids G that are locally compact, Hausdorff and étale, and have totally disconnected unit spaces. This puts us in the setting of [7].

Let

$$A(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$$

as in [7]. For $f, g \in A(G) \subseteq C_c(G)$, define

$$\begin{aligned} (1) \quad & f^*(\gamma) = \overline{f(\gamma^{-1})}; \text{ and} \\ (2) \quad & (f * g)(\gamma) = \sum_{r(\alpha)=r(\gamma)} f(\alpha)g(\alpha^{-1}\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta). \end{aligned}$$

Under these operations and pointwise addition and scalar multiplication, $A(G)$ is a complex $*$ -algebra. It coincides with the complex inverse semigroup algebra $\mathbb{C}G$ introduced in [33].² We call $A(G)$ the Steinberg algebra of G .

Theorem 4.1. *Let G be a locally compact, Hausdorff, étale groupoid such that $G^{(0)}$ is totally disconnected. Then $A(G)$ is simple if and only if both of the following conditions hold: the interior of the isotropy subgroupoid of G is $G^{(0)}$; and G is minimal in the sense that $G^{(0)}$ has no nontrivial open invariant subsets.*

Our proof was guided by that of Theorem 5.14 in [4]. However, their arguments rely heavily on the underlying k -graph structure so our approach looks very different. The first step is to prove that the Cuntz-Krieger uniqueness theorem for $A(G)$ [7, Theorem 5.2] still holds if we replace the hypothesis that G is topologically principal with the hypothesis that the interior of $\text{Iso}(G)$ is $G^{(0)}$.

Lemma 4.2. *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Suppose that the interior of the isotropy subgroupoid of G is $G^{(0)}$, and that π is a homomorphism of $A(G)$ such that $\ker(\pi) \neq \{0\}$. Then there is a compact open subset $V \subseteq G^{(0)}$ such that $\pi(1_V) = 0$.*

²We prefer the notation $A(G)$ because Steinberg's notation $\mathbb{C}G$ suggests the free \mathbb{C} -module with basis G , which is substantially larger. To avoid clashing with Steinberg's notation, we also use $\mathbb{F}(W)$ for the free complex module with basis W below.

Proof. Define $I = \ker(\pi)$. Fix $b \in I \setminus \{0\}$. Let $c := b^* * b$. For $u \in G^{(0)}$ we have

$$c(u) = \sum_{\gamma \in G_u} b^*(\gamma^{-1})b(\gamma) = \sum_{\gamma \in G_u} \overline{b(\gamma)}b(\gamma) \geq \max_{\gamma \in G_u} |b(\gamma)|^2.$$

In particular, the function

$$c_0 := \begin{cases} c(\gamma) & \text{if } \gamma \in G^{(0)}; \\ 0 & \text{otherwise} \end{cases}$$

is nonzero. Because $G^{(0)}$ is both open and closed, $c_0 \in A(G)$.

Using Lemma 3.6 of [7] we may write

$$c_0 = \sum_{U \in \mathcal{U}} a_U 1_U,$$

where \mathcal{U} is a collection of mutually disjoint, nonempty compact open subsets of $G^{(0)}$, and each a_U is nonzero. Let K be the support of $c - c_0$. Notice that $K \subseteq G \setminus G^{(0)}$.

Fix $U \in \mathcal{U}$. Since $\text{Iso}(G)^\circ = G^{(0)}$, the implication (1) \implies (4) of Lemma 3.1 implies that there exists an open set $V \subseteq U$ such that $VKV = \emptyset$. Since G has a basis of compact open sets, we can assume V is also compact.

For $\gamma \in G$ we have

$$(1_V(c - c_0)1_V)(\gamma) = 1_V(r(\gamma))(c - c_0)(\gamma)1_V(s(\gamma)) = 0.$$

So $1_V c 1_V = 1_V c_0 1_V = a_U 1_V$. Hence $1_V \in I$. □

Another key ingredient in our proof of Theorem 4.1 is the following generalisation of the *infinite-path representation* of a Kumjian-Pask algebra as defined on page 9 of [4]. In our setting, the infinite-path space becomes the unit space of G . In fact, the construction of [4] works for any invariant subset W of $G^{(0)}$. Given such a set W , we write $\mathbb{F}(W)$ for the free (complex) module with basis W . We use these representations to construct nontrivial ideals of $A(G)$ when there exists either a nontrivial open invariant subset of $G^{(0)}$ or a nonempty open subset of $\text{Iso}(G) \setminus G^{(0)}$.

Proposition 4.3. *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space and let W be an invariant subset of $G^{(0)}$.*

- (1) *For every compact open bisection $B \subseteq G$, there is a unique function $f_B : G^{(0)} \rightarrow \mathbb{F}(W)$ that has support contained in $s(B)$ and satisfies $f_B(s(\gamma)) = r(\gamma)$ for all $\gamma \in B$.*
- (2) *There is a unique representation $\pi_W : A(G) \rightarrow \text{End}(\mathbb{F}(W))$ such that $\pi_W(1_B)u = f_B(u)$ for every compact open bisection B and all $u \in W$.*

Proof. Let B be a compact, open bisection in G . The formula $s(\gamma) \mapsto r(\gamma)$ for γ in B specifies a well-defined homeomorphism from $s(B)$ to $r(B)$. Thus, the function f_B can be defined as stated in (1). To prove (2), first notice that the universal property of the free module $\mathbb{F}(W)$ implies that there is an element $t_B \in \text{End}(\mathbb{F}(W))$ extending f_B . Let $c : G \rightarrow \{e\}$ be the trivial cocycle. Then every bisection of G is e -graded under c , so the set $B_*^{\text{co}}(G)$ of [7, Definition 3.10] is the set of all compact open bisections of G . We claim that the collection $\{t_B : B \in B_*^{\text{co}}(G)\}$ gives a representation of $B_*^{\text{co}}(G)$ in $\text{End}(\mathbb{F}(W))$ as defined in Definition 3.10 of [7].

To prove our claim, we must verify that:

- (R1) $t_\emptyset = 0$;

- (R2) $t_B t_D = t_{BD}$ for all compact open bisections B and D ; and
- (R3) $t_B + t_D = t_{B \cup D}$ whenever B and D are disjoint compact open bisections such that $B \cup D$ is a bisection.

It is straightforward to check that each of these conditions holds for the functions f_B , and hence for the endomorphisms t_B as well.

Now, the universal property of $A(G)$, stated in Theorem 3.11 of [7], gives a unique homomorphism $\pi_W : A(G) \rightarrow \text{End}(\mathbb{F}(W))$ such that $\pi_W(1_B) = t_B$ for all $B \in B_*^{\text{co}}(G)$. The homomorphism π_W is nonzero because t_B is nonzero whenever $s(B) \cap W \neq \emptyset$. It satisfies $\pi_W(1_B)u = f_B(u)$ because each t_B extends f_B . \square

Proposition 4.4. *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and let $\pi := \pi_{G^{(0)}} : A(G) \rightarrow \text{End}(\mathbb{F}(G^{(0)}))$ be the homomorphism of Proposition 4.3. Then π is injective if and only if the interior of the isotropy subgroupoid of G is $G^{(0)}$.*

Proof. First suppose that the interior of the isotropy subgroupoid of G is $G^{(0)}$. Since $\pi(1_V) \neq 0$ for all compact open $V \subseteq G^{(0)}$, π is injective by the contrapositive of Lemma 4.2.

Now suppose that $\text{Iso}(G)^\circ \neq G^{(0)}$. By (1) \iff (3) of Proposition 3.1(3), there exists a nonempty compact open bisection $B \subseteq G \setminus G^{(0)}$ so that for every $\gamma \in B$, $r(\gamma) = s(\gamma)$. To see that $\ker(\pi)$ is non-trivial, notice that $B \neq s(B)$ but $f_B = f_{s(B)}$, where f_B is defined in Proposition 4.3(1). Thus $\pi_{G^{(0)}}(1_B) = \pi_{G^{(0)}}(1_{s(B)})$ giving $1_B - 1_{s(B)} \in \ker(\pi_{G^{(0)}})$. Since $B \neq s(B)$ we have $1_B - 1_{s(B)} \neq 0$, so $\ker(\pi_{G^{(0)}}) \neq \{0\}$. \square

Proposition 4.5. *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Then G is minimal in the sense that $G^{(0)}$ has no nontrivial open invariant subsets if and only if every nonzero $f \in A(G)$ such that $\text{supp } f \subseteq G^{(0)}$ generates $A(G)$ as an ideal.*

Proof. Suppose G is minimal. Fix $f \in A(G) \setminus \{0\}$ such that $\text{supp } f \subseteq G^{(0)}$. Let I be the ideal of $A(G)$ generated by f . Fix $g \in A(G)$; we must show that $g \in I$. Since f is nonzero and locally constant [7, Lemma 3.4], there exist $c \in \mathbb{C} \setminus \{0\}$ and a compact open bisection $U \subseteq G^{(0)}$ so that $f|_U \equiv c$. Then $1_U = \frac{1}{c} 1_U * f \in I$. Let $K := r(\text{supp}(g)) \subseteq G^{(0)}$. Then K is compact and open by [7, Lemma 3.2]. Since $s(G^U)$ is a nonempty open invariant set, it is all of $G^{(0)}$. Therefore $K \subseteq s(G^U)$. So for each $u \in K$, there exists γ_u with $r(\gamma_u) \in U$ and $s(\gamma_u) = u$. For each u , let B_u be a compact open bisection containing γ_u such that $r(B_u) \subseteq U$ and $s(B_u) \subseteq K$. Then $1_{s(B_u)} = 1_{B_u}^* * 1_U * 1_{B_u}$ belongs to I . Since K is compact, there is a finite subset $\{v_1, \dots, v_n\}$ of K such that $\{s(B_{v_i}) : 1 \leq i \leq n\}$ covers K . By disjointification of the collection $\{s(B_{v_i}) : 1 \leq i \leq n\}$ (see [7, Remark 2.5]), we may assume that the $s(B_{v_i})$ are mutually disjoint. For each i , the function $k_i := 1_{s(B_{v_i})}$ belongs to I , so $1_K = \sum_{i=1}^n k_i \in I$. Hence $g = 1_K * g \in I$.

Conversely, suppose G is not minimal. Let U be a nontrivial open invariant subset of $G^{(0)}$. Then the complement $W := G^{(0)} \setminus U$ is itself an invariant subset of $G^{(0)}$. Let $\pi_W : A(G) \rightarrow \text{End } \mathbb{F}(W)$ be the nonzero homomorphism of Proposition 4.3. The kernel of π_W is a proper ideal of $A(G)$. To complete the proof, it suffices to show that $\ker(\pi_W) \neq \{0\}$. To see this, let $B \subseteq U$ be a compact open set. Then $1_B \in \ker \pi_W \setminus \{0\}$. \square

Proof of Theorem 4.1. Suppose $A(G)$ is simple. Then $\pi_{G^{(0)}}$ is injective so Proposition 4.4 implies that $\text{Iso}(G)^\circ = G^{(0)}$. Since $A(G)$ is simple, every function with support contained in $G^{(0)}$ generates $A(G)$ as an ideal. Hence, G is minimal by Proposition 4.5.

Conversely, suppose that $\text{Iso}(G)^\circ = G^{(0)}$ and that G is minimal. Fix a nonzero ideal I in $A(G)$. Then I is the kernel of the quotient homomorphism $q : A(G) \rightarrow A(G)/I$. Lemma 4.2 implies that there is a compact open subset $V \subseteq G^{(0)}$ such that $q(1_V) = 0$. That is, $1_V \in I$. Proposition 4.5 implies that the ideal generated by 1_V is all of $A(G)$, so $I = A(G)$. \square

5. SIMPLICITY OF GROUPOID C^* -ALGEBRAS

For details of the following, see, for example, [27] or [23]. Let G be a second-countable, locally compact, Hausdorff, étale groupoid. The formulas (1) and (2) for convolution and involution on $A(G)$ described in the preceding section also define a convolution and involution on $C_c(G)$. With these operations, and pointwise addition and scalar multiplication, $C_c(G)$ a complex $*$ -algebra. The I -norm on $C_c(G)$ defined by

$$\|f\|_I = \sup_{u \in G^{(0)}} \max \left\{ \sum_{\gamma \in G_u} |f(\gamma)|, \sum_{\gamma \in G^u} |f(\gamma)| \right\}$$

is a $*$ -algebra norm (see Proposition II.1.4 of [27]) but not typically a C^* -norm. The full norm on $C_c(G)$ is defined by

$$\|f\| := \sup \{ \|\pi(f)\| : \pi \text{ is an } I\text{-norm-bounded } *\text{-representation of } C_c(G) \},$$

and $C^*(G)$ is defined to be the completion of $C_c(G)$ in the full norm.

There is a distinguished family of I -norm-bounded representations of $C_c(G)$, called the regular representations, indexed by the units of G and denoted Ind_u , $u \in G^{(0)}$. Specifically, for $u \in G^{(0)}$, the regular representation Ind_u is the representation of $C_c(G)$ on $\ell^2(G_u)$ implemented by convolution. That is, $\text{Ind}_u(f)\delta_\gamma = \sum_{\beta \in G^{r(\gamma)}} f(\beta^{-1}\gamma)\delta_\beta$. The reduced C^* -algebra $C_r^*(G)$ is the completion of $C_c(G)$ in the reduced norm $\|f\|_r = \sup_{u \in G^{(0)}} \|\text{Ind}_u(f)\|$. The reduced norm is dominated by the full norm, so $C_r^*(G)$ is a quotient of $C^*(G)$.

We can now state our main theorem.

Theorem 5.1. *Let G be a second-countable, locally compact, Hausdorff, étale groupoid. Then $C^*(G)$ is simple if and only if all of the following conditions are satisfied.*

- (1) $C^*(G) = C_r^*(G)$;
- (2) G is topologically principal in the sense that the units with trivial isotropy are dense in $G^{(0)}$; and
- (3) G is minimal in the sense that $G^{(0)}$ has no nontrivial open invariant subsets.

Our proof of Theorem 5.1 relies on the following adaptation of the augmentation representation of a discrete group. Let G be a groupoid as in Theorem 5.1. For each $u \in G^{(0)}$, let $[u]$ denote the orbit of u under G ; that is $[u] = r(G_u)$.

Proposition 5.2. *Let G be a second-countable, locally compact, Hausdorff, étale groupoid. Fix $u \in G^{(0)}$. There is a unique representation $\pi_{[u]}$ of $C^*(G)$ on $\ell^2([u]) = \overline{\text{span}} \{ \delta_v : v \in [u] \}$ such that for each $f \in C_c(G)$ and $v \in [u]$,*

$$(3) \quad \pi_{[u]}(f)\delta_v := \sum_{\gamma \in G_v} f(\gamma)\delta_{r(\gamma)}.$$

Proof. For $f \in C_c(G)$ and a finite linear combination $h = \sum_{v \in [u]} h_v \delta_v$, let $f \cdot h$ be the vector $\sum_{v \in [u]} \sum_{\gamma \in G_v} f(\gamma) h_v \delta_{r(\gamma)}$. Then $h \mapsto f \cdot h$ is linear, and $f \cdot \delta_v$ is equal to the right-hand side of (3). The following is adapted directly from the proof of [27, Proposition II.1.7]. Fix $f \in C_c(G)$. For $v, w \in [u]$, we have

$$(4) \quad (f \cdot \delta_v | \delta_w) = \sum_{\gamma \in G_v} (f(\gamma) \delta_{r(\gamma)} | \delta_w) = \sum_{\gamma \in G_v^w} f(\gamma) = \sum_{\gamma \in G_w} (\delta_v | \overline{f(\gamma^{-1})} \delta_w) = (\delta_v | f^* \cdot \delta_w).$$

Since $k \mapsto f \cdot k$ is linear on $\text{span}\{\delta_v : v \in [u]\}$, it follows that $(f \cdot k | k') = (k | f^* \cdot k')$ for all $k, k' \in C_c([u])$. In particular, for a finite linear combination $h = \sum_{v \in [u]} h_v \delta_v$,

$$\begin{aligned} \|f \cdot h\|^2 &= ((f^* f) \cdot h | h) \\ &= \left| \sum_{\gamma \in G_{[u]}} \overline{(f^* f)(\gamma)} h_{s(\gamma)} h_{r(\gamma)} \right| \\ &\leq \sum_{\gamma \in G_{[u]}} |(f^* f)(\gamma)| |h_{s(\gamma)}| |h_{r(\gamma)}| \\ &= \sum_{\gamma \in G_{[u]}} (|(f^* f)(\gamma)|^{1/2} |h_{s(\gamma)}|) (|(f^* f)(\gamma)|^{1/2} |h_{r(\gamma)}|) \end{aligned}$$

So the Cauchy-Schwarz inequality gives

$$\begin{aligned} \|f \cdot h\|^2 &\leq \left(\sum_{\gamma \in G_{[u]}} |(f^* f)(\gamma)| |h_{s(\gamma)}|^2 \right)^{1/2} \left(\sum_{\beta \in G_{[u]}} |(f^* f)(\beta)| |h_{r(\beta)}|^2 \right)^{1/2} \\ &= \left(\sum_{v \in [u]} \left(\sum_{\gamma \in G_v} |(f^* f)(\gamma)| \right) |h_v|^2 \right)^{1/2} \left(\sum_{w \in [u]} \left(\sum_{\beta \in G^w} |(f^* f)(\beta)| \right) |h_w|^2 \right)^{1/2} \\ &\leq \|(f^* f)\|_I \|h\|^2. \end{aligned}$$

Proposition II.1.4 of [27] (or direct calculation) shows that $\|f^* f\|_I \leq \|f\|_I^2$, and it follows that $\|f \cdot h\| \leq \|f\|_I \|h\|$.

Thus, for each $f \in C_c(G)$ the formula (3) determines a bounded linear operator $\pi_{[u]}(f)$ on $\ell^2([u])$, and the map $f \mapsto \pi_{[u]}(f)$ is bounded with respect to the I -norm. By definition of the norm on $C^*(G)$, it therefore remains only to show that $\pi_{[u]}$ is a $*$ -homomorphism from $C_c(G)$ to $\mathcal{B}(\ell^2([u]))$. The calculation (4) shows that $\pi_{[u]}(f)^* = \pi_{[u]}(f^*)$. For $f, g \in C_c(G)$ and $v \in [u]$,

$$\begin{aligned} \pi_{[u]}(f * g) \delta_v &= \sum_{\gamma \in G_v} (f * g)(\gamma) \delta_{r(\gamma)} = \sum_{\alpha \beta \in G_v} f(\alpha) g(\beta) \delta_{r(\alpha)} \\ &= \sum_{\beta \in G_v} \sum_{\alpha \in G_{r(\beta)}} f(\alpha) g(\beta) \delta_{r(\alpha)} = \sum_{\beta \in G_v} \pi_{[u]}(f)(g(\beta) \delta_{r(\beta)}) = \pi_{[u]}(f) \pi_{[u]}(g) \delta_v. \end{aligned}$$

Hence $\pi_{[u]}$ is a $*$ -homomorphism as required. \square

Remark 5.3. The direct sum $\epsilon_G := \bigoplus_{[u] \in G^{(0)}/G} \pi_{[u]}$ of G is faithful on $C_0(G^{(0)})$. To see this, fix $f \in C_c(G^{(0)}) \setminus \{0\}$ and $u \in G^{(0)}$ such that $f(u) \neq 0$. Then $\|\epsilon_G(f)\| \geq \|\pi_{[u]}(f) \delta_u\| = \|f(u) \delta_u\| \neq 0$. If G is a (discrete) group, then ϵ_G is just the 1-dimensional representation of $C^*(G)$ induced by the unitary representation $\epsilon : g \mapsto 1$ of G , sometimes called the *augmentation representation* of G .

Remark 5.4. Since we can construct the representations $\pi_{[u]}$ directly, there was no need to make use of Renault's theory of representations of G . However, we could have recovered Proposition 5.2 from the results of [22] as follows. There is a representation (in the sense of [22]) of G on the Borel Hilbert bundle $G^{(0)} \times \mathbb{C}$ determined by $U_\gamma : (s(\gamma), z) \mapsto (r(\gamma), z)$. If we view counting measure on $[u]$ as a measure $\mu_{[u]}$ on G with support $[u]$, then $\pi_{[u]}$ is equivalent to the representation R_μ of $C^*(G)$ obtained from [22, Proposition 7.1] applied to $(U, G^{(0)} \times \mathbb{C}, \mu_{[u]})$. For this equivalence, it is crucial that the modular function with respect to $\mu_{[u]}$ is the identity function. This is automatic because G is second-countable and étale so the orbit is countable.

Proposition 5.5. *Let G be a second-countable, locally compact, Hausdorff, étale groupoid.*

- (1) *Suppose that G is topologically principal in the sense that the set of units with trivial isotropy is dense in $G^{(0)}$. Then every ideal I of the reduced C^* -algebra $C_r^*(G)$ satisfies $I \cap C_c(G^{(0)}) \neq \{0\}$.*
- (2) *Suppose that every ideal of the full C^* -algebra $C^*(G)$ satisfies $I \cap C_c(G^{(0)}) \neq \{0\}$. Then G is topologically principal.*

Proof. (1) Fix a nonzero ideal I of $C_r^*(G)$. We must show that $I \cap C_c(G^{(0)}) \neq \{0\}$. The first two paragraphs of our proof are almost identical to the start of the proof of [18, Lemma 3.5] (which, in turn, acknowledges [2, Proposition 2.4]). Let $P : C_r^*(G) \rightarrow C_0(G^{(0)})$ be the faithful conditional expectation extending restriction of compactly supported functions [27, Proposition II.4.8]. Fix $a \in I^+$ such that $\|P(a)\| = 1$, and fix $b \in C_r^*(G)^+ \cap C_c(G)$ such that $\|a - b\| < 1/4$. Then $\|P(b)\| > 3/4$ and $b - P(b)$ has compact support $K \subseteq G \setminus G^{(0)}$.

Let $U := \{u \in G^{(0)} : b(u) > 3/4\}$. Lemmas 3.1 and 3.3 imply that there exists a nonempty open set $V \subseteq U$ such that $VKV = \emptyset$. Fix a nonempty open W whose closure is contained in V .

Since G is normal, there is a function $f \in C_c(G^{(0)})$ such that $W \subseteq \text{supp}(f) \subseteq V$. For $\gamma \in G$

$$(f * (b - P(b)) * f)(\gamma) = f(r(\gamma))(b - P(b))(\gamma)f(s(\gamma)) = 0.$$

So $fbf = fP(b)f \geq 3/4f^2$. Therefore

$$faf \geq fbf - 1/4f^2 \geq 1/2f^2.$$

Since I is an ideal, it is hereditary. Since a , and hence faf , belongs to I , it follows that $1/2f^2 \in I \cap C_c(G^{(0)}) \setminus \{0\}$.

(2) We prove the contrapositive. Suppose that G is not topologically principal. Lemmas 3.3 and 3.1 imply that there is an open bisection B in $G \setminus G^{(0)}$ consisting entirely of isotropy. Let ϵ_G be the direct sum representation defined in Remark 5.3. We show that $\ker(\epsilon_G)$ is a nontrivial ideal in $C^*(G)$ that does not intersect $C_0(G^{(0)})$. By Remark 5.3, $\ker(\epsilon_G) \cap C_0(G^{(0)}) = \{0\}$ so it suffices to construct a nonzero element of $\ker \epsilon_G$.

For each $u \in s(B)$, let γ_u be the unique element in B such that $s(\gamma_u) = u$. Fix a nonzero function $f \in C_c(G)$ such that $\text{supp}(f) \subseteq B$, and define $f_0 \in C_c(G^{(0)})$ by

$$f_0(u) := \begin{cases} f(\gamma_u) & \text{if } u \in s(B), \\ 0 & \text{otherwise.} \end{cases}$$

Since $B \cap G^{(0)} = \emptyset$ and since $f \neq 0$, we have $f - f_0 \neq 0$. We claim that $\epsilon_G(f - f_0) = 0$; that is, $\pi_{[u]}(f - f_0) = 0$ for all $u \in G^{(0)}$. To see this, fix $u \in G^{(0)}$ and $v \in [u]$. Then

$$\pi_{[u]}(f - f_0)\delta_v = \sum_{\gamma \in G_v} f(\gamma)\delta_{r(\gamma)} - \sum_{\alpha \in G_v} f_0(\alpha)\delta_{r(\alpha)}.$$

If $v \notin s(B)$, then $f(\gamma) = f_0(\alpha) = 0$ for all $\gamma, \alpha \in G_v$, so $\pi_{[u]}(f - f_0)\delta_v = 0$. Suppose that $v \in s(B)$. Since f_0 is supported on units and f is supported on B ,

$$\sum_{\gamma \in G_v} f(\gamma)\delta_{r(\gamma)} - \sum_{\alpha \in G_v} f_0(\alpha)\delta_{r(\alpha)} = f(\gamma_v)\delta_{r(\gamma_v)} - f_0(v)\delta_v = f(\gamma_v)\delta_{r(\gamma_v)} - f(\gamma_v)\delta_v.$$

Since $B \subseteq \text{Iso}(G)$, we have $r(\gamma_v) = s(\gamma_v) = v$, and it follows that $\pi_{[u]}(f - f_0)\delta_v = 0$. \square

We will use the following standard lemma in the proofs of Proposition 5.7 and Corollary 5.9.

Lemma 5.6. *Let G be a locally compact, Hausdorff, étale groupoid. Suppose that $h \in C_c(G)$ is supported on a bisection B and that $f \in C_c(G^{(0)})$. Then $h * f * h^* \in C_c(G^{(0)})$ with support contained in $r(B) \subseteq G^{(0)}$ and satisfies*

$$(h * f * h^*)(r(\gamma)) = |h(\gamma)|^2 f(s(\gamma)) \quad \text{for all } \gamma \in B.$$

Proof. For $\alpha \in G$, we have

$$(5) \quad h * f * h^*(\alpha) = \sum_{\gamma\eta\beta^{-1}=\alpha} h(\gamma)f(\eta)\overline{h(\beta)}.$$

Fix $\gamma\eta\beta^{-1} \in G$ with $h(\gamma)f(\eta)\overline{h(\beta)} \neq 0$. Since $\text{supp}(f) \subseteq G^{(0)}$, we have $\eta = s(\gamma) = s(\beta)$. Since h is supported on the bisection B , it follows that $\gamma, \beta F \in B$ and $\beta = \gamma$. Hence $\gamma\eta\beta^{-1} = \gamma s(\gamma)\gamma^{-1} = r(\gamma) \in r(B)$. Thus the sum on the right of (5) is zero if $\alpha \notin r(B)$, and has only one nonzero term $h(\gamma)f(s(\gamma))\overline{h(\gamma)} = |h(\gamma)|^2 f(s(\gamma))$ if $\alpha = r(\gamma) \in r(B)$. \square

Proposition 5.7. *Let G be a second-countable, locally compact, Hausdorff, étale groupoid. The following are equivalent:*

- (1) G is minimal in the sense that $G^{(0)}$ has no nontrivial open invariant subsets;
- (2) the ideal of $C^*(G)$ generated by any nonzero $f \in C_c(G^{(0)})$ is $C^*(G)$; and
- (3) the ideal of $C_r^*(G)$ generated by any nonzero $f \in C_c(G^{(0)})$ is $C_r^*(G)$.

Proof. (1) \implies (2) and (1) \implies (3). Let $f \in C_c(G^{(0)}) \setminus \{0\}$ and let I be the ideal of $C^*(G)$ generated by f . We claim that $C_c(G^{(0)}) \subseteq I$. Since $I \cap C_0(G^{(0)})$ is an ideal of $C_0(G^{(0)})$, it suffices to show that for each $u \in G^{(0)}$, there exists $g \in I \cap C_0(G^{(0)})$ such that $g(u) \neq 0$. Fix $u \in G^{(0)}$. Let $U := \{v \in G^{(0)} : f(v) \neq 0\}$. Then U is nonempty and open, and hence $r(G_U)$ is open because s is continuous and the local homeomorphism r is an open map. So $r(G_U)$ is a nonempty open invariant set, and hence is equal to $G^{(0)}$ because G is minimal. In particular, there exists $\gamma \in G$ such that $s(\gamma) \in U$ and $r(\gamma) = u$. Fix $h \in C_c(G)$ such that $\text{supp}(h)$ is contained in a bisection and $h(\gamma) = 1$. Lemma 5.6 implies that $(h * f * h^*)(u) = |h(\gamma)|^2 f(s(\gamma)) = f(s(\gamma)) \neq 0$. So $g := h * f * h^*$ belongs to $I \cap C_0(G^{(0)})$ with $g(u) = 1$. This proves the claim.

Fix $F \in C_c(G)$. Then any $g \in C_c(G^{(0)})$ such that $g|_{r(\text{supp}(F))} \equiv 1$ satisfies $g * F = F$. Hence $C_c(G) \subseteq I$, and so $I = C^*(G)$. Let $q : C^*(G) \rightarrow C_r^*(G)$ be the quotient map. Then the ideal I_r of $C_r^*(G)$ generated by f is $q(I)$. Since q restricts to the identity map on $C_c(G)$, we have $C_c(G) \subseteq I_r$ as well, and hence $I_r = C_r^*(G)$.

(2) \implies (1) and (3) \implies (1). We prove the contrapositive. Suppose that U is a nonempty proper open invariant subset of $G^{(0)}$. Fix $f \in C_c(G) \setminus \{0\}$ such that $\text{supp}(f) \subseteq U$. Then $f \in C_c(G^{(0)})$. Fix $u \in G^{(0)} \setminus U$. Since $G^{(0)} \setminus U$ is invariant, $[u] \subseteq G^{(0)} \setminus U$, so $f(v) = 0$ for all $v \in [u]$. It follows that the image of f under the regular representation Ind_u is zero. On the other hand, for any $g \in C_c(G^{(0)})$ such that $g(u) = 1$, we have $\text{Ind}_u(g)\delta_u = g(u)\delta_u \neq 0$. So Ind_u is a nonzero representation of $C_c(G)$ with nontrivial kernel. Since Ind_u extends to each of $C_r^*(G)$ and $C^*(G)$ it follows that the ideals of each of $C^*(G)$ and $C_r^*(G)$ generated by f are proper ideals. \square

Remark 5.8. Suppose that G is locally compact, Hausdorff and étale. Thomsen observes in [34] that if G has a unit with trivial isotropy, then G is topologically principal whenever it is minimal (see Remark 2.2). He then deduces that if G has a unit with trivial isotropy, then $C^*(G)$ is simple if and only if G is minimal. We recover this result from Proposition 5.5(2) together with (1) \iff (3) of Proposition 5.7.

Proof of Theorem 5.1. Suppose $C^*(G)$ is simple. Then the quotient map from $C^*(G) \rightarrow C_r^*(G)$ has trivial kernel and hence the two coincide. Moreover, $C^*(G)$ is the only nonzero ideal of $C^*(G)$ and $C^*(G) \cap C_0(G^{(0)}) \neq \{0\}$ so Proposition 5.5 implies that G is topologically principal. The simplicity of $C^*(G)$ implies that every $f \in C_c(G^{(0)})$ generates $C^*(G)$ as an ideal and so Proposition 5.7 implies that G is minimal.

Now suppose that $C^*(G) = C_r^*(G)$ and that G is topologically principal and minimal. Fix a nonzero ideal I in $C^*(G)$. Since $C^*(G) = C_r^*(G)$, Proposition 5.5(1) implies there exists a nonzero $f \in C_c(G^{(0)}) \cap I$; and then (1) \implies (3) of Proposition 5.7 implies that the ideal generated by f is $C^*(G)$. Thus $I = C^*(G)$. \square

Corollary 5.9 below characterises the measurewise-amenable, étale groupoids for which the ideal structure of $C^*(G)$ coincides with the G -invariant ideal structure of $C_0(G^{(0)})$. The argument for the “if” implication is standard (see, for example, [27, Proposition 4.6]), but we include it for completeness.

The notion of amenability for groupoids is somewhat technical; for a detailed discussion, see [3]. For our purposes, we only need the following two facts. First, if G is measurewise amenable, then $C^*(G) = C_r^*(G)$ [3, Proposition 3.3.5]. Second, suppose that $U \subseteq G^0$ is open and invariant. If G is measurewise amenable then each of G_U and $G_{G^{(0)} \setminus U}$ is measurewise amenable [3, Corollary 5.3.21].³

If $D \subseteq G^0$ is a closed invariant set, then $\{f \in C_c(G) : f|_{G_D} \equiv 0\}$ is an ideal of $C^*(G)$ isomorphic to $C^*(G_{G^{(0)} \setminus D})$, and the quotient is isomorphic to $C^*(G_D)$ (see [21, Lemma 2.10]). This decomposition fails in general for reduced C^* -algebras.

Corollary 5.9. *Let G be a second-countable, locally compact, Hausdorff groupoid. Suppose that G is measurewise amenable and étale. Then $D \mapsto \overline{\{f \in C_c(G) : f|_{G_D} \equiv 0\}}$ is a bijection between closed invariant subsets of $G^{(0)}$ and ideals of $C^*(G)$ if and only if, for every closed invariant $D \subseteq G^{(0)}$, G_D is topologically principal.*

Proof. First, we claim that there is a bijection between closed invariant subsets D and ideals of the form $I \cap C_0(G^{(0)})$, where I is an ideal in $C^*(G)$. Let D be a closed invariant subset. Then the map that sends D to the ideal $\{f \in C_0(G^{(0)}) : f|_D \equiv 0\} \subseteq C_0(G^{(0)})$ is a well defined injection. To see that this map is a surjection onto the set of ideals of the form $I \cap C_0(G)$, let I be an ideal of $C^*(G)$. Since the multiplication in $C_0(G^{(0)})$ is pointwise,

³ G_U embeds properly into G since G acts properly on itself.

the ideal $I \cap C_0(G^{(0)})$ has the form $\{f \in C_0(G^{(0)}) : f|_D \equiv 0\}$ for some closed $D \subseteq G^{(0)}$. We show that D is invariant by establishing that its complement is invariant. Fix $\gamma \in G$ such that $s(\gamma) \notin D$, and $f \in I \cap C_0(G^{(0)})$ such that $f(s(\gamma)) = 1$. We must show that $r(\gamma) \notin D$. Let B be an open bisection of G containing γ , and h be a function supported on B such that $h(\gamma) = 1$. By Lemma 5.6, $(h * f * h^*)(r(\gamma)) = |h(\gamma)|^2 f(s(\gamma)) = 1$, so $r(\gamma) \notin D$. This proves our claim.

Now, it suffices to show that $I \mapsto I \cap C_0(G^{(0)})$ is a bijection if and only if G_D is topologically principal for each closed invariant $D \subseteq G^{(0)}$.

First, suppose that G_D is topologically principal for every closed invariant $D \subseteq G^{(0)}$. Fix an ideal I of $C^*(G)$. Let J be the ideal of $C^*(G)$ generated by $I \cap C_c(G^{(0)})$. Then $J \subseteq I$. Since G is measurewise amenable, $C^*(G) = C_r^*(G)$. Hence $J \neq \{0\}$ by Proposition 5.5. We must show that $J = I$.

Let $J_0 := J \cap C_0(G^{(0)})$, and let $D := \{u \in G^{(0)} : f(u) = 0 \text{ for all } f \in J_0\}$; so J_0 is the ideal $\{f \in C_0(G^{(0)}) : f(u) = 0 \text{ for all } u \in D\}$ of $C_0(G^{(0)})$. As above, D is a closed invariant subset of $G^{(0)}$. So [29, Remark 4.10] implies that restriction of functions induces an isomorphism $C^*(G)/J \cong C^*(G_D)$, and this isomorphism carries I/J to an ideal of $C^*(G_D)$ which has trivial intersection with $C_0(D)$ by construction of J . Corollary 5.3.21 of [3] implies that G_D is measurewise amenable, so Proposition 5.5 implies that I/J is trivial and hence $I = J$ as required.

We prove the reverse implication by contrapositive. Suppose that there exists a closed invariant subset D of $G^{(0)}$ such that G_D is not topologically principal. Lemma 3.3 shows that $\text{Iso}(G_D)^\circ \neq D$. Let $I(D)$ be the ideal of $C^*(G)$ generated by $\{f \in C_c(G^{(0)}) : f|_D \equiv 0\}$. Again by [29, Remark 4.10], restriction of functions induces an isomorphism $\phi : C^*(G)/I(D) \rightarrow C^*(G_D)$. Proposition 5.5 applied to the groupoid G_D gives a nontrivial ideal J of $C^*(G_D)$ such that $J \cap C_c(D) = \{0\}$. Let $q_D : C^*(G) \rightarrow C^*(G)/I(D)$ and $q_J : C^*(G_D) \rightarrow C^*(G_D)/J$ be the quotient maps. Let $K := \ker(q_J \circ \phi \circ q_D)$. That $J \cap C_c(D) = \{0\}$ forces $K \cap C_c(G^{(0)}) = C_0(D)$. That J is nontrivial implies that $K \neq I(D)$. Since $K \cap C_0(G^{(0)}) = I(D) \cap C_0(G^{(0)})$, the result follows. \square

Remark 5.10. The hypothesis of measurewise amenability in Corollary 5.9 is required only to guarantee that $C^*(G_D) = C_r^*(G_D)$ for every closed invariant subset of $G^{(0)}$. So the theorem also holds under this formally weaker (but less checkable) hypothesis.

Recall that an étale groupoid G is *locally contracting* if for every nonempty open subset U of $G^{(0)}$, there exists an open subset V of U and an open bisection B such that $\overline{V} \subseteq s(B)$ and $r(B\overline{V}) \subsetneq V$ [2, Definition 2.1]. In the following corollary, we use Theorem 5.1 and Lemma 3.1 to strengthen [2, Proposition 2.4].

Corollary 5.11. *Let G be a second-countable, locally compact, Hausdorff groupoid. Suppose that G is also locally contracting and étale, and that $C^*(G)$ is simple. Then $C^*(G)$ is purely infinite.*

Proof. Theorem 5.1 implies that G is topologically principal, so [2, Proposition 2.4] implies that every nonzero hereditary $*$ -subalgebra of $C^*(G)$ contains an infinite projection. \square

6. EXAMPLES

In this section, we present some examples to indicate why the hypotheses on our main theorem are needed. We also demonstrate that the second assertion of Lemma 3.3 fails if G is either not second countable or not étale.

Example 6.1 (Amenability). One might wonder at first sight whether Theorem 5.1 could be strengthened to a characterisation of simplicity for $C_r^*(G)$ for locally compact, Hausdorff, étale groupoids. We cannot expect this: the free group \mathbb{F}_2 on two generators, regarded as a discrete groupoid with just one unit, is a second countable, locally compact, Hausdorff, étale groupoid such that $\text{Iso}(\mathbb{F}_2)^\circ = \mathbb{F}_2 \neq \{1_{\mathbb{F}_2}\}$, and Powers proved in [24] that $C_r^*(\mathbb{F}_2)$ is simple.

Example 6.2 (Twisted groupoid algebras). Our characterisation of simplicity does not extend to groupoid C^* -algebras that are ‘twisted’ by a 2-cocycle, as defined in [27]. To see why, consider the group \mathbb{Z}^2 regarded as a discrete groupoid with one unit. This is a locally compact, Hausdorff, étale, amenable groupoid with $\text{Iso}(\mathbb{Z}^2) = \mathbb{Z}^2$, so our theorem reduces to the observation that $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$ is not simple. To see that this does not extend to twisted algebras, fix $\theta \in [0, 1] \setminus \mathbb{Q}$ and let $\phi_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{T}$ be the \mathbb{T} -valued 2-cocycle $\theta((m_1, m_2), (n_1, n_2)) = e^{i\theta(m_2 n_1)}$. It is well known that the twisted groupoid C^* -algebra $C^*(\mathbb{Z}^2, \phi_\theta)$ is the irrational rotation algebra A_θ and hence simple.

We proved in Lemma 3.3 that if G is second-countable and étale then G is topologically principal if and only if the interior of the isotropy is $G^{(0)}$. We also showed that the ‘only if’ implication holds without the assumption of second-countability. One might ask whether the conditions are equivalent in general. Of course, if G is not étale then $G^{(0)}$ is usually not open, so we cannot expect the interior of the isotropy to be equal to $G^{(0)}$. But an equivalent condition for étale groupoids which does generalise to non-étale groupoids is that interior of the isotropy is contained in $G^{(0)}$. The next example shows that for non-étale G , this condition is strictly weaker than G being topologically principal.

Example 6.3. Let $X := (0, 1) \times \mathbb{T}$. Define a continuous action of \mathbb{R} on X by

$$t \cdot (s, e^{i\theta}) = (s, e^{i(\theta + 2st\pi)}).$$

Let G be the transformation-group groupoid $X \rtimes \mathbb{R}$. For each $u = (s, e^{i\theta}) \in G^{(0)}$, the isotropy group is $G_u^u = \{u\} \times \frac{1}{s}\mathbb{Z}$, so no point in $G^{(0)}$ has trivial isotropy. Fix an open set U in G . We must show that $U \setminus \text{Iso}(G) \neq \emptyset$. Since U is open, there exist $0 < a < b < 1$, $\theta \in (0, 2\pi)$, and $t \in \mathbb{R} \setminus \{0\}$ such that $((a, b) \times \{e^{i\theta}\}) \times \{t\} \subseteq U$. Fix $s \in (a, b)$. If $st \notin \mathbb{Z}$ then $((s, e^{i\theta}), t) \in U \setminus \text{Iso}(G)$. So suppose that $st \in \mathbb{Z}$. Choose $\varepsilon \in (0, \frac{1}{t})$ such that $s + \varepsilon \in (a, b)$. Then $st < (s + \varepsilon)t < st + 1$, so $(s + \varepsilon)t \notin \mathbb{Z}$. Hence $((s + \varepsilon, e^{i\theta}), t) \in U \setminus \text{Iso}(G)$.

Our next example also satisfies $\text{Iso}(G)^\circ = G^{(0)}$ without being topologically principal. This time G is étale and has totally disconnected unit space, but is not second countable. This shows that Lemma 4.2 is strictly stronger than [7, Theorem 5.2].

Example 6.4. Let K denote the Cantor set and give \mathbb{T} the discrete topology. Let X be the topological product space $(K \cap (0, 1)) \times \mathbb{T}$. Define an (algebraic) action of \mathbb{R} on X by restriction of the action of Example 6.3. Endow the acting copy of \mathbb{R} with the discrete topology. Then the action is continuous and the transformation groupoid G is étale (but not second countable). Moreover, every open subset of G which does not intersect $G^{(0)}$ contains a subset of the form $((K \cap (a, b)) \times \{e^{i\theta}\}) \times \{t\}$ as in Example 6.3, so arguing as in that example (using that $K \cap (a, b)$ has no isolated points), we see that the interior of the isotropy subgroupoid is $G^{(0)}$.

In Examples 6.3 and 6.4, there are many nontrivial closed proper invariant subsets of $G^{(0)}$. We have not found an example of a locally compact, Hausdorff, étale groupoid that is

also minimal and satisfies the equivalent conditions of Lemma 3.1 but is not topologically principal.

7. EXEL-VERSHIK SYSTEMS

When we first began trying to prove that $A(G)$ is simple if and only if G is minimal and topologically principal, we went looking for examples — other than k -graph groupoids — of étale groupoids with totally disconnected unit spaces to test the hypothesis. We were led to the work of Exel and Vershik in [11]. Their characterisation of simplicity [11, Theorem 11.2] led us to condition (3) of Lemma 3.1 and from there to our main simplicity theorems. In this section, we investigate the relationship between our result and that of Exel and Vershik. We obtain a generalisation of their simplicity theorem to a very broad class of dynamical systems.

Recall that an *Ore semigroup* is a monoid M which is cancellative and satisfies:

$$(6) \quad \text{for all } m, n \in M, \text{ there exist } p, q \in M \text{ such that } pm = qn.$$

Definition 7.1. An *Exel-Vershik system* is a triple (X, M, T) consisting of a second-countable, locally compact, Hausdorff space X , a countable discrete Ore semigroup M , and an action T of M on X by local homeomorphisms; we write T^m for the local homeomorphism associated to $m \in M$.

Remark 7.2. Every commutative monoid and every group is an Ore semigroup: if M is commutative then $nm = mn$ and if M is a group then $m^{-1}m = n^{-1}n$. Indeed, a monoid M is an Ore semigroup if and only if there is an embedding of M in a group $\Gamma = \Gamma(M)$ such that $\Gamma = M^{-1}M$ (see, for example, [19, Theorem 1.2]). The group Γ is unique up to isomorphism and we call it the *Grothendieck group of M* .

Let (X, M, T) be an Exel-Vershik system and let $\Gamma(M)$ be the Grothendieck group of M . Consider the set

$$G(X, T) := \{(x, m^{-1}n, y) \in X \times \Gamma(M) \times X : m, n \in M, T^m(x) = T^n(y)\}.$$

Remark 7.2 and [10, Proposition 3.1] imply that the formulas

$$\begin{aligned} r(x, m^{-1}n, y) &= x & s(x, m^{-1}n, y) &= y \\ (x, m^{-1}n, y) \cdot (y, p^{-1}q, z) &:= (x, m^{-1}np^{-1}q, z) & (x, m^{-1}n, y)^{-1} &= (y, n^{-1}m, x) \end{aligned}$$

make $G(X, T)$ into a groupoid.⁴

For precompact open subsets U, V of X and $m, n \in M$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms and $T^m(U) = T^n(V)$, let

$$Z(U, V, m, n) := \{(x, m^{-1}n, y) : x \in U, y \in V, T^m(x) = T^n(y)\}.$$

Then the sets $Z(U, V, m, n)$ form a base of precompact open bisections for a second-countable topology on $G(X, T)$. Under this topology, $G(X, T)$ is a locally compact, Hausdorff, étale groupoid [10, Proposition 3.2].

Examples 7.3. (1) If $M = \mathbb{N}$ then $G(X, T)$ is the Deaconu-Renault groupoid of the local homeomorphism T [8].

⁴We define $G(X, T)$ slightly differently than in [10] to make it more compatible with Example 7.3(2) below.

- (2) Let M be a discrete group and suppose T is an action of M on X . Then T^g is a homeomorphism for all $g \in M$ so in particular a local homeomorphism. The Grothendieck group of M is M . Further if $T^m(x) = T^n(y)$ then $x = T^{m^{-1}n}(y)$. So

$$G(X, T) = \{(T^g(y), g, y) : y \in X, g \in M\}$$

and for each basic open set $Z(U, V, m, n)$, we have

$$\begin{aligned} Z(U, V, m, n) &= \{(T^{m^{-1}n}(y), m^{-1}n, y) : y \in V, T^{m^{-1}n}(y) \in U\} \\ &= \{(T^{m^{-1}n}(y), m^{-1}n, y) : y \in V \cap T^{n^{-1}m}(U)\}. \end{aligned}$$

Thus the map $(T^g(y), g, y) \mapsto (g, y)$ induces an isomorphism of $G(X, T)$ with the transformation-group groupoid $X \rtimes_T M$.

- (3) Let Λ be a row-finite k -graph with no sources as in [17]. Recall that Λ^∞ denotes the infinite-path space of Λ and that for $n \in \mathbb{N}^k$ we write σ^n for the shift map $\sigma^n(x)(p, q) = x(p + n, q + n)$ on Λ^∞ . The groupoid G_Λ of [17] is then identical to the groupoid corresponding to the Exel-Vershik system $(\Lambda^\infty, \mathbb{N}^k, \sigma)$. Kumjian and Pask show that G_Λ is amenable in [17, Theorem 5.5].

The following definition is a straightforward generalisation of [11, Definition 10.1] to arbitrary Exel-Vershik systems. The notion of topological freeness of (X, M, T) given here is formally weaker than that of [5, Definition 1] when M is a countable discrete abelian group.

Definition 7.4. We say an Exel-Vershik system (X, M, T) is *topologically free* if for every pair $m \neq n \in M$ the set $\{x \in X : T^m(x) = T^n(x)\}$ has empty interior.

Proposition 7.5. *An Exel-Vershik system (X, M, T) is topologically free if and only if the associated groupoid $G(X, T)$ is topologically principal.*

Proof. Suppose that (X, M, T) is not topologically free. Then there exist $m \neq n \in M$ and an open set $U \subseteq X$ such that $T^m(x) = T^n(x)$ for all $x \in U$. Fix $z \in U$ and neighbourhoods W_m and W_n of z in U such that $T^l|_{W_l}$ is a homeomorphism for $l = m, n$. Define $V = W_m \cap W_n$. Then $z \in V$ and $T^l|_V$ is a homeomorphism for both $l = m, n$. Since $T^m(x) = T^n(x)$ for all $x \in V \subseteq U$ we have $T^m(V) = T^n(V)$, so the set $Z(V, V, m, n)$ is an open subset of $\text{Iso}(G(X, T)) \setminus X$. Thus Lemma 3.3 implies that $G(X, T)$ is not topologically principal.

Conversely, suppose that G is not topologically principal. By Lemma 3.3, there exists an open bisection $B \subseteq G(X, T) \setminus X$ such that $r(\gamma) = s(\gamma)$ for all $\gamma \in B$. So there is a basic open set $Z(U, V, m, n)$ contained in B . That $B \subseteq G(X, T) \setminus X$ forces $m \neq n$. Since $Z(U, V, m, n) \subseteq B$ and $r(\gamma) = s(\gamma)$ for all $\gamma \in B$, we have $U = V$ and $T^m x = T^n x$ for all $x \in U$. So (X, M, T) is not topologically free. \square

Remark 7.6. The following special case of Example (1) was considered in [6]. Let X be a compact Hausdorff space and $T : X \rightarrow X$ a covering map. Proposition 7.5 implies that (X, M, T) is topologically principal, and so Proposition 5.5 recovers [6, Theorem 6 ((1) \Leftrightarrow (2))].

Remark 7.7. Recall from [17, Definition 4.3] that a row-finite k -graph Λ with no sources is *aperiodic* if for any $v \in \Lambda^0$ there exists $x \in \Lambda^\infty$ such that $r(x) = v$ and $\sigma^n(x) \neq \sigma^m(x)$ for all $m \neq n \in \mathbb{N}^k$. Recall also from [32, Definition 1] that Λ has *no local periodicity* if for any $n \neq m \in \mathbb{N}^k$ and $v \in \Lambda^0$ there exists $x \in \Lambda^\infty$ such that $r(x) = v$ and $\sigma^n(x) \neq \sigma^m(x)$. Kumjian and Pask show that Λ is aperiodic if and only if G_Λ is topologically principal [17,

Proposition 4.5]. A similar argument shows that Λ has no local periodicity if and only if the Exel-Vershik system $(\Lambda^\infty, \mathbb{N}^k, \sigma)$ is topologically free. Thus Proposition 7.5 can be viewed as a generalisation of [32, Lemma 3.2].

Theorem 5.1 and Proposition 7.5 imply that if the full and reduced C^* -algebras of the groupoid $G(X, T)$ of an Exel-Vershik system (X, M, T) coincide, then the associated C^* -algebra $C(X) \rtimes_T \Gamma(M)$ is simple if and only if the system is topologically free and for each $x \in X$ the orbit

$$[x]_T := \{y \in X : T^m y = T^n x \text{ for some } m, n \in M\}$$

is dense in X . It is therefore an interesting question whether $C^*(G(X, T)) = C_r^*(G(X, T))$ whenever $\Gamma(M)$ is amenable. We give a partial answer which applies to all systems for which Exel and Renault's results guarantee that the Exel crossed product $C(X) \rtimes_T \Gamma(M)$ of [9] coincides with $C^*(G(X, T))$.

Corollary 7.8. *Suppose M is an Ore semigroup such that $\Gamma(M)$ is amenable. Suppose that (X, M, T) is an Exel-Vershik system satisfying the standing hypotheses 4.1 of [10]. Then $C^*(G(X, T)) = C_r^*(G(X, T))$. Moreover, $C(X) \rtimes_T \Gamma(M)$ is simple if and only if the system is topologically free and $\overline{[x]}_T = X$ for each $x \in X$.*

Proof. The second assertion follows from Theorem 5.1 once we show that $C^*(G(X, T)) = C_r^*(G(X, T))$. For this let π be the isomorphism $\pi : C(X) \rtimes_T \Gamma(M) \cong C^*(G(X, T))$ of [10, Theorem 6.6], and let $q : C^*(G(X, T)) \rightarrow C_r^*(G(X, T))$ be the quotient map. It suffices to show that $q \circ \pi$ is injective. For this, just run the proof of [10, Theorem 6.6] replacing $C^*(G(X, T))$ with $C_r^*(G(X, T))$. It is only necessary to check that $q \circ (\pi \times \sigma)$ is injective on each graded subspace, and for this the argument of [10, Proposition 6.5] suffices because the calculations in that proof involve elements of $C_c(G(X, T))$. \square

Amenability issues disappear when considering the Steinberg algebras of Section 4.1. So Exel-Vershik systems (X, M, T) where X is a Cantor set should provide many interesting examples of Steinberg algebras $A(G(X, T))$ for which simplicity is characterised by Theorem 4.1.

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